Noncommutative localization in algebraic L-theory

Citation for published version:
Ranicki, A 2009, 'Noncommutative localization in algebraic L-theory', *Advances in Mathematics*, vol. 220, no. 3, pp. 894-912. https://doi.org/10.1016/j.aim.2008.10.003

Digital Object Identifier (DOI):
10.1016/j.aim.2008.10.003

Published In:
Advances in Mathematics

NONCOMMUTATIVE LOCALIZATION IN ALGEBRAIC L-THEORY

ANDREW RANICKI

ABSTRACT. Given a noncommutative (Cohn) localization $A \to \sigma^{-1}A$ which is injective and stably flat we obtain a lifting theorem for induced f.g. projective $\sigma^{-1}A$-module chain complexes and localization exact sequences in algebraic L-theory, matching the algebraic K-theory localization exact sequence of Neeman-Ranicki [3] and Neeman [2].

CONTENTS

INTRODUCTION

The series of papers [3], [2], studied the algebraic K-theory of the noncommutative (Cohn) localization $\sigma^{-1}A$ of a ring A inverting a collection σ of morphisms of f.g. projective left A-modules. By definition, $\sigma^{-1}A$ is stably flat if

$$\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) \ = \ 0 \ (i \geq 1) \ .$$

An (A, σ)-module is an A-module T which admits a f.g. projective A-module resolution

$$0 \longrightarrow P \xrightarrow{\ s\ } Q \longrightarrow T \longrightarrow 0$$

with $s : \sigma^{-1}P \to \sigma^{-1}Q$ an isomorphism of the induced $\sigma^{-1}A$-modules. For $A \longrightarrow \sigma^{-1}A$ which is injective and stably flat we obtained an algebraic K-theory localization exact sequence

$$\cdots \to K_n(A) \to K_n(\sigma^{-1}A) \to K_{n-1}(H(A, \sigma)) \to K_{n-1}(A) \to \dots$$

with $H(A, \sigma)$ the exact category of (A, σ)-modules.

Let C be a bounded $\sigma^{-1}A$-module chain complex such that each $C_i = \sigma^{-1}P_i$ is induced from a f.g. projective A-module P_i. The *chain complex lifting problem* is to decide if C is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A-modules. The problem has a trivial affirmative solution for a commutative or Ore localization, by

Key words and phrases. noncommutative localization, chain complexes, L-theory.

the clearing of denominators, when C is actually isomorphic to $\sigma^{-1}D$. In general, it is not possible to lift chain complexes: the injective noncommutative localizations $A \to \sigma^{-1}A$ which are not stably flat constructed in Neeman, Ranicki and Schofield [4, Remark 2.13] provide examples of induced f.g. projective $\sigma^{-1}A$-module chain complexes of dimensions $\geqslant 3$ which cannot be lifted.

In §1 we solve the chain complex lifting problem in the injective stably flat case, obtaining the following results (Theorems 1.4,1.5) :

Theorem 0.1. *For a stably flat injective noncommutative localization $A \to \sigma^{-1}A$ every bounded chain complex C of induced f.g. projective $\sigma^{-1}A$-modules is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A-modules. Moreover, if C is n-dimensional*

$$C \ : \ \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0 \to \ldots$$

then D can be chosen to be n-dimensional. □

In §2 we consider the algebraic L-theory of a noncommutative localization, obtaining the following results (Theorems 2.4, 2.5, 2.9) :

Theorem 0.2. *Let $A \longrightarrow \sigma^{-1}A$ be a noncommutative localization of a ring with involution A, such that σ is invariant under the involution.*
(i) *There is a localization exact sequence of quadratic L-groups*

$$\cdots \longrightarrow L_n(A) \longrightarrow L^I_n(\sigma^{-1}A) \xrightarrow{\partial} L_n(A,\sigma) \longrightarrow L_{n-1}(A) \longrightarrow \cdots$$

with $I = \mathrm{im}(K_0(A) \longrightarrow K_0(\sigma^{-1}A))$, and $L_n(A,\sigma)$ the cobordism group of $\sigma^{-1}A$-contractible $(n-1)$-dimensional quadratic Poincaré complexes over A.
(ii) *If $\sigma^{-1}A$ is stably flat over A there is a localization exact sequence of symmetric L-groups*

$$\cdots \longrightarrow L^n(A) \longrightarrow L^n_I(\sigma^{-1}A) \xrightarrow{\partial} L^n(A,\sigma) \longrightarrow L^{n-1}(A) \longrightarrow \cdots$$

with $L^n(A,\sigma)$ the cobordism group of $\sigma^{-1}A$-contractible $(n-1)$-dimensional symmetric Poincaré complexes over A.
(iii) *If $A \longrightarrow \sigma^{-1}A$ is injective then $L^n(A,\sigma)$ (resp. $L_n(A,\sigma)$) is the cobordism group of n-dimensional symmetric (resp. quadratic) Poincaré complexes of (A,σ)-modules.* □

The L-theory exact sequences of Theorem 0.2 for an injective Ore localization $A \longrightarrow \sigma^{-1}A$ (which is flat and hence stably flat) were obtained in Ranicki [5]. The quadratic L-theory exact sequence of 0.2 (i) for arbitrary injective $A \longrightarrow \sigma^{-1}A$ was obtained by Vogel [8], [9]. The symmetric L-theory exact sequence of 0.2 (ii) is new.

We refer to [6, 7] for some of the applications of the algebraic L-theory of noncommutative localizations to topology.

Amnon Neeman used to be a coauthor of the paper, but decided to withdraw in May 2007.

1. Lifting chain complexes

If $A \longrightarrow \sigma^{-1}A$ is a stably flat localization, we know from [3, Theorem 0.4, Proposition 4.5 and Theorem 3.7] that the functor $Ti : \frac{D^{\text{perf}}(A)}{\mathcal{R}^c} \longrightarrow D^{\text{perf}}(\sigma^{-1}A)$ is just an idempotent completion; it is fully faithful and all objects in $D^{\text{perf}}(\sigma^{-1}A)$ are, up to isomorphisms, direct summands of objects in the image of Ti. A fairly easy consequence of this is the following. Let $C \in D^{\text{perf}}(\sigma^{-1}A)$ be the complex

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0,$$

with C^i all finitely generated, projective A–modules. Then there is complex $X \in D^{\text{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L\otimes_A X$. That is, C is homotopy equivalent to the tensor product with $\sigma^{-1}A$ of a perfect complex over the ring A. In Section 1 we prove this (Theorem 1.4), and then refine the result to show that X may be chosen to be a complex of the form

$$0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0\ .$$

(Proof in Theorem 1.5).

Remark 1.1. The proof of Theorem 1.4 relies on the following fact about triangulated categories. Suppose $\mathcal{A}$ is a full, triangulated subcategory of a triangulated category $\mathcal{B}$, and suppose all objects in $\mathcal{B}$ are direct summands of objects of $\mathcal{A}$. An object $X \in \mathcal{B}$ belongs to $\mathcal{A} \subset \mathcal{B}$ if and only if $[X] \in K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$. This fact may be found, for example, in [1, Proposition 4.5.11], but for the reader's convenience its proof is included here in Lemma 1.2 and Proposition 1.3.

□

We begin by reminding the reader of some basic facts about Grothendieck groups. For any additive category $\mathcal{A}$ we define $K_0^{\text{add}}(\mathcal{A})$ to be the Grothendieck group of the split exact category $\mathcal{A}$. This means that the short exact sequences in $\mathcal{A}$ are precisely the split sequences. It is well known that every element of $K_0^{\text{add}}(\mathcal{A})$ can be expressed as

$$[X] - [Y]$$

for X and Y objects of $\mathcal{A}$. The expressions $[X]-[Y]$ and $[X']-[Y']$ are equal in $K_0^{\text{add}}(\mathcal{A})$ if and only if there exists an object $P \in \mathcal{A}$ and an isomorphism

$$X \oplus Y' \oplus P \;=\; X' \oplus Y \oplus P.$$

If $\mathcal{A}$ happens to be a triangulated category, then $K_0(\mathcal{A})$ means the quotient of $K_0^{\text{add}}(\mathcal{A})$ by a subgroup we will denote $T(\mathcal{A})$. The subgroup $T(\mathcal{A})$ is defined as the group generated by all

$$[X] - [Y] + [Z],$$

where there exists a distinguished triangle in $\mathcal{A}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

We prove:

Lemma 1.2. *Suppose $\mathcal{B}$ is a triangulated category. Let $\mathcal{A}$ be a full, triangulated subcategory of $\mathcal{B}$. Assume further that every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

Then the map $f : K_0^{\mathrm{add}}(\mathcal{A}) \longrightarrow K_0^{\mathrm{add}}(\mathcal{B})$ induces a surjection $T(\mathcal{A}) \longrightarrow T(\mathcal{B})$. In symbols: $f\big(T(\mathcal{A})\big) = T(\mathcal{B})$.

Proof. Let $[X]-[Y]+[Z]$ be a generator of $T(\mathcal{B}) \subset K_0^{\mathrm{add}}(\mathcal{B})$. We need to show it lies in the image of $T(\mathcal{A}) \subset K_0^{\mathrm{add}}(\mathcal{A})$. Suppose therefore that

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a distinguished triangle in $\mathcal{B}$. Because every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A}$, we can choose objects C and D with

$$X \oplus C, \qquad Z \oplus D$$

both lying in $\mathcal{A}$. But then we have a two distinguished triangles in $\mathcal{B}$

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ C & \longrightarrow & C \oplus D & \longrightarrow & D & \xrightarrow{0} & \Sigma C \end{array}$$

and their direct sum is a distinguished triangle

$$X \oplus C \longrightarrow Y \oplus C \oplus D \longrightarrow Z \oplus D \longrightarrow \Sigma(X \oplus C).$$

Two of the objects lie in $\mathcal{A}$. Since the subcategory $\mathcal{A} \subset \mathcal{B}$ is full and triangulated, the entire distinguished triangle lies in $\mathcal{A}$. Thus

$$[X \oplus C] - [Y \oplus C \oplus D] + [Z \oplus D] \quad = \quad [X] - [Y] + [Z]$$

lies in the image of $T(\mathcal{A})$. $\square$

The next proposition is well-known; again, the proof is included for the convenience of the reader.

Proposition 1.3. *Let the hypotheses be as in Lemma 1.2. That is, suppose $\mathcal{B}$ is a triangulated category. Let $\mathcal{A}$ be a full, triangulated subcategory of $\mathcal{B}$. Assume further that every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

If X is an object of $\mathcal{B}$ and $[X]$ lies in the image of the natural map $f : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$, then $X \in \mathcal{A}$.

Proof. If we consider $[X]$ as an element of $K_0^{\mathrm{add}}(\mathcal{B})$, then saying that its image in $K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$ is equivalent to saying that, modulo $T(\mathcal{B})$, $[X]$ lies in the image of $K_0^{\mathrm{add}}(\mathcal{A})$. That is,

$$[X] \in T(\mathcal{B}) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) \subset K_0^{\mathrm{add}}(\mathcal{B}).$$

By Lemma 1.2 we have that $f\big(T(\mathcal{A})\big) = T(\mathcal{B})$. Thus

$$\begin{array}{rcl} T(\mathcal{B}) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) & = & f\big(T(\mathcal{A})\big) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) \\ & = & f\big(K_0^{\mathrm{add}}(\mathcal{A})\big). \end{array}$$

That means there exist objects C and D in $\mathcal{A} \subset \mathcal{B}$ and an identity in $K_0^{\mathrm{add}}(\mathcal{B})$

$$[X] = [C] - [D].$$

There must therefore be an object $P \in \mathcal{B}$ and an isomorphism

$$X \oplus D \oplus P \;\simeq\; C \oplus P.$$

But P is an object of $\mathcal{B}$, hence a direct summand of an object of $\mathcal{A}$. There is an object $P' \in \mathcal{B}$ with $P \oplus P' \in \mathcal{A}$. We have an isomorphism

$$X \oplus D \oplus P \oplus P' \;\simeq\; C \oplus P \oplus P'.$$

Putting $D' = D \oplus P \oplus P$ and $C' = C \oplus P \oplus P'$ we have objects C', D' in $\mathcal{A}$, and a (split) distinguished triangle

$$D' \longrightarrow C' \longrightarrow X \longrightarrow \Sigma D'.$$

Since $\mathcal{A} \subset \mathcal{B}$ is triangulated we conclude that $X \in \mathcal{A}$. □

The relevance of these results to our work here is

Theorem 1.4. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose we are given a perfect complex C over $\sigma^{-1}A$. Suppose further that $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ is of the form*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0$$

where each C^i is a finitely generated, projective A–module. Then C is homotopy equivalent to $\{\sigma^{-1}A\}^L\!\otimes_A X$, for some $X \in D^{\mathrm{perf}}(A)$.

Proof. The localization is stably flat. By [3, Theorem 0.4] the functor $T : \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$ is an equivalence of categories. By [3, Proposition 4.5 and Theorem 3.7] we also know that the functor $i : \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} \longrightarrow \mathcal{T}^c$ is fully faithful, and that every object in $\mathcal{T}^c$ is isomorphic to a direct summand of an object in the image of i. Next we apply Proposition 1.3, with $\mathcal{B} = D^{\mathrm{perf}}(\sigma^{-1}A)$ and $\mathcal{A}$ the full subcategory containing all objects isomorphic to $Ti(x)$, for any $x \in \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}$.

Now C is an object of $D^{\mathrm{perf}}(\sigma^{-1}A)$, and in $K_0\big(D^{\mathrm{perf}}(\sigma^{-1}A)\big)$ we have an identity

$$[C] = \sum_{\ell=-\infty}^{\infty} (-1)^\ell [\sigma^{-1}C^\ell]$$

with

$$[\sigma^{-1}C^\ell] = [\{\sigma^{-1}A\} \otimes_A C^\ell] = [TiC^\ell]$$

certainly lying in the image of the map

$$K_0(Ti) : K_0\left(\frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}\right) \longrightarrow K_0\big(D^{\mathrm{perf}}(\sigma^{-1}A)\big).$$

Proposition 1.3 therefore tells us that C is isomorphic to an object in the image of the functor Ti. There exists a perfect complex $X \in D^{\mathrm{perf}}(A)$ and a homotopy equivalence $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. □

The problem with Theorem 1.4 is that it gives us no bound on the length of the complex X with $\{\sigma^{-1}A\}^L \otimes_A X \simeq C$. We really want to know

Theorem 1.5. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ is the complex*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

Then the complex $X \in D^{\mathrm{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$, whose existence is guaranteed by Theorem 1.4, may be chosen to be a complex

$$0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0\ .$$

If $m = n$ this is easy. For $m < n$ we need to prove something. Our proof will appeal to the results of [3, Section 4]. We remind the reader that this was the section which dealt with the subcategories $\mathcal{K}[m,n]$ of complexes in $\mathcal{R}^c$ vanishing outside the range $[m,n]$. First we need a lemma.

Lemma 1.6. *Let M and N be any finitely generated projective A–modules. We may view M and N as objects in the derived category $D^{\mathrm{perf}}(A)$, concentrated in degree 0. Then any map in $\mathcal{T}^c(\pi M, \pi N)$ can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\mathrm{perf}}(A)$ as below*

$$M \xrightarrow{\ \beta\ } Y \xleftarrow{\ \alpha\ } N\ .$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\ \alpha\ } Y \longrightarrow \Sigma X$$

and X may be chosen to lie in $\mathcal{K}[0,1]$.

Proof. By [3, Proposition 4.5 and Theorem 3.7] we know that the map

$$i : \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} \longrightarrow \mathcal{T}^c$$

is fully faithful. Therefore

$$\mathcal{T}^c(\pi M, \pi N) \quad = \quad \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}(M, N).$$

That is, any map $\pi M \longrightarrow \pi N$ can be written as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\mathrm{perf}}(A)$ as below

$$M \xrightarrow{\ \beta\ } Y \xleftarrow{\ \alpha\ } N\ .$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \xrightarrow{\beta} \Sigma X$$

and X may be chosen to lie in $\mathcal{R}^c$. What is not clear is that we may choose X in $\mathcal{K}[0,1] \subset \mathcal{R}^c$.

The easy observation is that we may certainly modify our choice of X to lie in $\mathcal{K} \subset \mathcal{R}^c$. This follows from [2, Lemma 4.5], which tells us that for any choice of X as above there exists an X' with $X \oplus X'$ isomorphic to an object in $\mathcal{K}$. We have a distinguished triangle

$$X \oplus X' \longrightarrow N \xrightarrow{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} Y \oplus \Sigma X' \xrightarrow{\beta \oplus 1} \Sigma(X \oplus X')$$

and a diagram

$$M \xrightarrow{\begin{pmatrix} \beta \\ 0 \end{pmatrix}} Y \oplus \Sigma X' \xleftarrow{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} N\ ,$$

and replacing our original choices by these we may assume $X \in \mathcal{K}$. Now we have to shorten X.

By [2, Lemma 4.7], there exists a triangle in $\mathcal{R}^c$

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[1,\infty)$ and $X'' \in \mathcal{K}(-\infty,1]$. The composite $X' \longrightarrow X \longrightarrow N$ is a map from $X' \in \mathcal{K}[1,\infty)$ to $N \in \mathcal{S}^{\leq 0}$, which must vanish. Hence we have that $X \longrightarrow N$ factors as $X \longrightarrow X'' \longrightarrow N$. We complete to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & N \arrow[r, "\alpha"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "\gamma"'] & \Sigma X \arrow[d] \\
X'' \arrow[r] & N \arrow[r, "\gamma\alpha"] & Y'' \arrow[r] & \Sigma X''
\end{tikzcd}$$

and another representative of our morphism is the diagram

$$M \xrightarrow{\gamma\beta} Y'' \xleftarrow{\gamma\alpha} N$$

We may, on replacing Y by Y'', assume $X \in \mathcal{K}(-\infty,1]$.

Applying [2, Lemma 4.7] again, we have that any $X \in \mathcal{K}(-\infty,1]$ admits a triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[0,1]$ and $X'' \in \mathcal{K}(-\infty,0]$. Form the octahedron

$$\begin{array}{ccccccc}
X' & \longrightarrow & N & \xrightarrow{\alpha'} & Y' & \longrightarrow & \Sigma X' \\
\big\downarrow & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle \gamma}\big\downarrow & & \big\downarrow \\
X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\
 & & & & \big\downarrow & & \big\downarrow \\
 & & & & \Sigma X'' & \xrightarrow{1} & \Sigma X''
\end{array}$$

The composite $M \longrightarrow Y \longrightarrow \Sigma X''$ is a map from the projective module M, viewed as a complex concentrated in degree 0, to $\Sigma X'' \in \mathcal{K}(\infty,-1]$. This composite must vanish. The map $\beta : M \longrightarrow Y$ therefore factors as $M \xrightarrow{\beta'} Y' \xrightarrow{\gamma} Y$, and our morphism in $\mathcal{T}^c$ has a representative

$$M \xrightarrow{\beta'} Y' \xleftarrow{\alpha'} N$$

so that in the triangle

$$X' \longrightarrow N \xrightarrow{\alpha'} Y' \longrightarrow \Sigma X'$$

X' may be chosen to lie in $\mathcal{K}[0,1]$. $\square$

Now we are ready for

Proof of Theorem 1.5. We are given a complex $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ of the form

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

To eliminate the trivial case, assume $m \le n+1$. Shifting, we may assume $m = 0$ and $n \ge 1$. Theorem 1.4 guarantees that C is homotopy equivalent to $\{\sigma^{-1}A\}^L\otimes_A D$, with $D \in D^{\mathrm{perf}}(A)$. But D need not be supported on the interval $[0,n]$. We need to show how to shorten D. Assume therefore that D is supported on $[-1,n]$. We will show how to replace D by a complex supported on $[0,n]$. Shortening a complex supported on $[0,n+1]$ is dual, and we leave it to the reader.

We may suppose therefore that $D \in D^{\mathrm{perf}}(A)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow \cdots \longrightarrow D^n \longrightarrow 0 \longrightarrow \cdots$$

and that there is a homotopy equivalence of $\sigma^{-1}D$ with a shorter complex, that is a commutative diagram

$$\begin{array}{ccccccccccccc}
\longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \cdots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow \\
 & \big\downarrow & & \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}C^0 & \longrightarrow & \cdots & \longrightarrow & \sigma^{-1}C^n & \longrightarrow & 0 & \longrightarrow \\
 & \big\downarrow & & \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & \\
\longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \cdots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow
\end{array}$$

so that the composite is homotopic to the identity. In particular, there is a map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$ so that $d\partial : \sigma^{-1}D^{-1} \longrightarrow \sigma^{-1}D^{-1}$ is the identity.

By [2, Proposition 3.1] the map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$ lifts uniquely to a map $d' : \pi D^0 \longrightarrow \pi D^{-1}$. By Lemma 1.6 the map d' can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, where α and β are, respectively, the chain maps

$$\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^{-1} & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow
\end{array}$$

and

$$\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^{0} & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & {\scriptstyle g}\downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow
\end{array}$$

The fact that $\sigma^{-1}\alpha$ is an equivalence tells us that the map $\sigma^{-1}r : \sigma^{-1}X \longrightarrow \sigma^{-1}Y$ is injective, with cokernel $\sigma^{-1}D^{-1}$. The fact that $\alpha^{-1}\beta$ agrees with d' means that the composite

$$\sigma^{-1}D^0 \xrightarrow{\sigma^{-1}g} \sigma^{-1}Y \longrightarrow \mathrm{Coker}(\sigma^{-1}r)$$

is just the map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$. Let X be the chain complex

$$\longrightarrow 0 \longrightarrow D^0 \oplus X \xrightarrow{\begin{pmatrix}\partial & 0\\ g & r\end{pmatrix}} D^1 \oplus Y \longrightarrow \cdots \longrightarrow D^n \longrightarrow 0 \longrightarrow$$

Let $f : X \longrightarrow D$ be the natural map of chain complexes

$$\begin{array}{ccccccccccc}
\longrightarrow 0 & \longrightarrow & 0 & \longrightarrow & D^0 \oplus X & \xrightarrow{\begin{pmatrix}\partial & 0\\ g & r\end{pmatrix}} & D^1 \oplus Y & \longrightarrow & \cdots & \longrightarrow & D^n \longrightarrow 0 \longrightarrow \\
 & & \downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow \\
\longrightarrow 0 & \longrightarrow & D^{-1} & \longrightarrow & D^0 & \xrightarrow[\partial]{} & D^1 & \longrightarrow & \cdots & \longrightarrow & D^n \longrightarrow 0 \longrightarrow
\end{array}$$

where the vertical maps labelled π_1 are the projections to the first factor of the direct sum. The map $\sigma^{-1}f$ is easily seen to be homotopy equivalence. Thus $\sigma^{-1}X$ is homotopy equivalent to $\sigma^{-1}D \cong C$. □

2. Algebraic L-theory

An *involution* on a ring A is an anti-automorphism

$$A \longrightarrow A \; ; \; r \mapsto \overline{r} \; .$$

The involution is used to regard a left A-module M as a right A-module by

$$M \times A \longrightarrow M \; ; \; (x, r) \mapsto \overline{r}x \; .$$

The *dual* of a (left) A-module M is the A-module

$$M^* \;=\; \mathrm{Hom}_A(M,A)\ ,\ A\times M^* \longrightarrow M^*\ ;\ (r,f)\mapsto (x\mapsto f(x)\overline{r})\ .$$

The *dual* of an A-module morphism $s:P\longrightarrow Q$ is the A-module morphism

$$s^*\ :\ Q^*\longrightarrow P^*\ ;\ f\mapsto (x\mapsto f(s(x)))\ .$$

If M is f.g. projective then so is M^*, and

$$M\longrightarrow M^{**}\ ;\ x\mapsto (f\mapsto \overline{f(x)})$$

is an isomorphism which is used to identify $M^{**}=M$.

Hypothesis 2.1. *In this section, we assume that*

(i) *A is a ring with involution,*
(ii) *the duals of morphisms $s:P\longrightarrow Q$ in σ are morphisms $s^*:Q^*\longrightarrow P^*$ in σ,*
(iii) *$\epsilon\in A$ is a central unit such that $\overline{\epsilon}=\epsilon^{-1}$ (e.g. $\epsilon=\pm 1$).*

The noncommutative localization $\sigma^{-1}A$ is then also a ring with involution, with $\epsilon\in\sigma^{-1}A$ a central unit such that $\overline{\epsilon}=\epsilon^{-1}$. □

We review briefly the chain complex construction of the f.g. projective ϵ-quadratic L-groups $L_*(A,\epsilon)$ and the ϵ-symmetric L-groups $L^*(A,\epsilon)$. Given an A-module chain complex C let the generator $T\in\mathbb{Z}_2$ act on the $\mathbb{Z}$-module chain complex $C\otimes_A C$ by the ϵ-transposition duality

$$T_\epsilon\ :\ C_p\otimes_A C_q\longrightarrow C_q\otimes_A C_p\ :\ x\otimes y\mapsto (-1)^{pq}\epsilon y\otimes x\ .$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W\ :\ \dots\longrightarrow \mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T}\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1+T}\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T}\mathbb{Z}[\mathbb{Z}_2]\ .$$

The *ϵ-symmetric* (resp. *ϵ-quadratic*) *Q-groups* of C are the $\mathbb{Z}_2$-hypercohomology (resp. $\mathbb{Z}_2$-hyperhomology) groups of $C\otimes_A C$

$$\begin{aligned}Q^n(C,\epsilon)\ &=\ H^n(\mathbb{Z}_2;C\otimes_A C)\ =\ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C\otimes_A C))\ ,\\ Q_n(C,\epsilon)\ &=\ H_n(\mathbb{Z}_2;C\otimes_A C)\ =\ H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C\otimes_A C))\ .\end{aligned}$$

The Q-groups are chain homotopy invariants of C. There are defined forgetful maps

$$\begin{aligned}&1+T_\epsilon\ :\ Q_n(C,\epsilon)\longrightarrow Q^n(C,\epsilon)\ ;\ \psi\mapsto (1+T_\epsilon)\psi\ ,\\ &Q^n(C,\epsilon)\longrightarrow H_n(C\otimes_A C)\ ;\ \phi\mapsto\phi_0\ .\end{aligned}$$

For f.g. projective C the function

$$C\otimes_A C\longrightarrow \mathrm{Hom}_A(C^*,C)\ ;\ x\otimes y\mapsto (f\mapsto \overline{f(x)}y)$$

is an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes, with $T\in\mathbb{Z}_2$ acting on $\mathrm{Hom}_A(C^*,C)$ by $\theta\mapsto\epsilon\theta^*$. The element $\phi_0\in H_n(C\otimes_A C)=H_n(\mathrm{Hom}_A(C^*,C))$ is a chain homotopy class of A-module chain maps $\phi_0:C^{n-*}\longrightarrow C$.

An *n-dimensional ϵ-symmetric complex over A* (C,ϕ) is a bounded f.g. projective A-module chain complex C together with an element $\phi \in Q^n(C,\epsilon)$. The complex (C,ϕ) is *Poincaré* if the A-module chain map $\phi_0 : C^{n-*} \longrightarrow C$ is a chain equivalence.

Example 2.2. A 0-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over A is essentially the same as a nonsingular ϵ-symmetric form (M,λ) over (A,σ), with $M = (C_0)^*$ a f.g. projective A-module and

$$\lambda \;=\; \phi_0 \;:\; M \times M \longrightarrow A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^* \;;\; x \mapsto (y \mapsto \lambda(x,y))$$

is an A-module isomorphism.

□

See pp. 210–211 of [6] for the notion of an *ϵ-symmetric (Poincaré) pair*. The *boundary* of an n-dimensional ϵ-symmetric complex (C,ϕ) is the $(n-1)$-dimensional ϵ-symmetric Poincaré complex

$$\partial(C,\phi) \;=\; (\partial C, \partial\phi)$$

with $\partial C = C(\phi_0 : C^{n-*} \longrightarrow C)_{*+1}$ and $\partial\phi$ as defined on p. 218 of [6]. The *n-dimensional ϵ-symmetric L-group* $L^n(A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes (C,ϕ) over A with C n-dimensional. In particular, $L^0(A,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric forms over A.

An n-dimensional ϵ-symmetric complex (C,ϕ) over A is *$\sigma^{-1}A$-Poincaré* if the $\sigma^{-1}A$-module chain map $\sigma^{-1}\phi_0 : \sigma^{-1}C^{n-*} \longrightarrow \sigma^{-1}C$ is a chain equivalence, in which case $\sigma^{-1}(C,\phi)$ is an n-dimensional ϵ-symmetric Poincaré complex over $\sigma^{-1}A$.

The *n-dimensional ϵ-symmetric Γ-group* $\Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric $\sigma^{-1}A$-Poincaré complexes (C,ϕ) over A such that $\sigma^{-1}C$ is chain equivalent to an n-dimensional induced f.g. projective $\sigma^{-1}A$-module chain complex. The *n-dimensional ϵ-symmetric L-group* $L^n(A,\sigma,\epsilon)$ is the cobordism group of $(n-1)$-dimensional ϵ-symmetric Poincaré complexes over A (C,ϕ) such that C is $\sigma^{-1}A$-contractible, i.e. $\sigma^{-1}C \simeq 0$.

Similarly in the ϵ-quadratic case, with groups $L_n(A,\epsilon)$, $\Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon)$, $L_n(A,\sigma,\epsilon)$. The ϵ-quadratic L- and Γ-groups are 4-periodic

$$\begin{aligned}
&L_n(A,\epsilon) \;=\; L_{n+2}(A,-\epsilon) \;=\; L_{n+4}(A,\epsilon)\;,\\
&\Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \;=\; \Gamma_{n+2}(A \longrightarrow \sigma^{-1}A,-\epsilon) \;=\; \Gamma_{n+4}(A \longrightarrow \sigma^{-1}A,\epsilon)\;,\\
&L_n(A,\sigma,\epsilon) \;=\; L_{n+2}(A,\sigma,-\epsilon) \;=\; L_{n+4}(A,\sigma,\epsilon)\;.
\end{aligned}$$

Proposition 2.3. *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ there is defined a localization exact sequence of ϵ-symmetric L-groups*

$$\cdots \longrightarrow L^n(A,\epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \cdots .$$

Similarly in the ϵ-quadratic case, with an exact sequence

$$\cdots \longrightarrow L_n(A,\epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots .$$

Proof. The relative group of $L^n(A,\epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric $\sigma^{-1}A$-Poincaré pairs over A $(f : C \longrightarrow D, (\delta\phi,\phi))$ with (C,ϕ) Poincaré. The effect of algebraic surgery on (C,ϕ) using this pair is a cobordant $(n-1)$-dimensional ϵ-symmetric Poincaré complex (C',ϕ') with C' $\sigma^{-1}A$-contractible. The function $(f : C \longrightarrow D, (\delta\phi,\phi)) \mapsto (C',\phi')$ defines an isomorphism between the relative group and $L^n(A,\sigma,\epsilon)$. □

Define

$$I \;=\; \mathrm{im}(K_0(A) \longrightarrow K_0(\sigma^{-1}A)) \;,$$

the subgroup of $K_0(\sigma^{-1}A)$ consisting of the projective classes of the f.g. projective $\sigma^{-1}A$-modules induced from f.g. projective A-modules. By definition, $L^n_I(\sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes over $\sigma^{-1}A$ (B,θ) such that $[B] \in I$. There are evident morphisms of Γ- and L-groups

$$\begin{aligned}
\sigma^{-1}\Gamma^* \;&:\; \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon) \;;\; (C,\phi) \mapsto \sigma^{-1}(C,\phi) \;, \\
\sigma^{-1}\Gamma_* \;&:\; \Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L_n^I(\sigma^{-1}A,\epsilon) \;;\; (C,\psi) \mapsto \sigma^{-1}(C,\psi) \;.
\end{aligned}$$

In general, the morphisms $\sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ need not be isomorphisms, since a bounded f.g. projective $\sigma^{-1}A$-module chain complex D with $[D] \in I$ need not be chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective A-module chain complex C.

It was proved in Chapter 3 of Ranicki [5] that if $A \longrightarrow \sigma^{-1}A$ is an injective Ore localization then the morphisms $\sigma^{-1}Q^*, \sigma^{-1}Q_*$, $\sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ are isomorphisms, so that there are defined localization exact sequences for both the ϵ-symmetric and the ϵ-quadratic L-groups

$$\begin{aligned}
&\cdots \longrightarrow L^n(A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \cdots \;, \\
&\cdots \longrightarrow L_n(A,\epsilon) \longrightarrow L_n^I(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots .
\end{aligned}$$

Special cases of these sequences were obtained by Milnor-Husemoller, Karoubi, Pardon, Smith, Carlsson-Milgram.

Let $G\pi : D(A) \to D(A)$ be the functor of Proposition 6.1 of [3], with $D(A)$ the derived category of A. For any bounded f.g. projective A-module chain complex C the natural A-module chain map

$$\varinjlim_{(B,\beta)} B \;=\; G\pi(C) \longrightarrow \sigma^{-1}C$$

induces morphisms

$$\begin{aligned} \sigma^{-1}Q^* &: \varinjlim_{(B,\beta)} Q^n(B,\epsilon) = Q^n(G\pi(C),\epsilon) \longrightarrow Q^n(\sigma^{-1}C,\epsilon) , \\ \sigma^{-1}Q_* &: \varinjlim_{(B,\beta)} Q_n(B,\epsilon) = Q_n(G\pi(C),\epsilon) \longrightarrow Q_n(\sigma^{-1}C,\epsilon) \end{aligned}$$

with the direct limits taken over all the bounded f.g. projective A-module chain complexes B with a chain map $\beta : C \longrightarrow B$ such that $\sigma^{-1}\beta : \sigma^{-1}C \longrightarrow \sigma^{-1}B$ is a $\sigma^{-1}A$-module chain equivalence. The natural projection $D \otimes_A D \longrightarrow D \otimes_{\sigma^{-1}A} D$ is an isomorphism for any bounded f.g. projective $\sigma^{-1}A$-module chain complex D (since this is already the case for $D = \sigma^{-1}A$), so the Q-groups of $\sigma^{-1}C$ are the same whether $\sigma^{-1}C$ is regarded as an A-module or $\sigma^{-1}A$-module chain complex.

Theorem 2.4. (Vogel [9], Theorem 8.4) *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ the morphisms*

$$\sigma^{-1}\Gamma_* : \Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L^I_n(\sigma^{-1}A,\epsilon) ; (C,\psi) \mapsto \sigma^{-1}(C,\psi)$$

are isomorphisms, and there is a localization exact sequence of ϵ-quadratic L-groups

$$\dots \longrightarrow L_n(A,\epsilon) \longrightarrow L^I_n(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \dots .$$

Proof. By algebraic surgery below the middle dimension it suffices to consider only the special cases $n = 0, 1$. In effect, it was proved in [9] that $\sigma^{-1}Q_*$ is an isomorphism for 0- and 1-dimensional C. □

It was claimed in Proposition 25.4 of Ranicki [6] that $\sigma^{-1}\Gamma^*$ is also an isomorphism, assuming (incorrectly) that the chain complex lifting problem can always be solved. However, we do have :

Theorem 2.5. *If $\sigma^{-1}A$ is a noncommutative localization of a ring with involution A which is stably flat over A, there is a localization exact sequence of ϵ-symmetric L-groups*

$$\dots \longrightarrow L^n(A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \dots .$$

Proof. For any bounded f.g. projective A-module chain complex C the natural A-module chain map $G\pi(C) \longrightarrow \sigma^{-1}C$ induces isomorphisms in homology

$$H_*(G\pi(C)) \cong H_*(\sigma^{-1}C) .$$

Thus the natural $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$G\pi(C) \otimes_A G\pi(C) \longrightarrow \sigma^{-1}C \otimes_A \sigma^{-1}C = \sigma^{-1}C \otimes_{\sigma^{-1}A} \sigma^{-1}C$$

induces isomorphisms of ϵ-symmetric Q-groups

$$\sigma^{-1}Q^* : \varinjlim_{(B,\beta)} Q^n(B,\epsilon) \longrightarrow Q^n(\sigma^{-1}C,\epsilon)$$

(and also isomorphisms $\sigma^{-1}Q_*$ of ϵ-quadratic Q-groups). By Theorem 0.1 every n-dimensional induced f.g. projective $\sigma^{-1}A$-module chain complex D is chain equivalent to $\sigma^{-1}C$ for an n-dimensional f.g. projective A-module chain complex C, with

$$Q^n(D,\epsilon) \;=\; Q^n(\sigma^{-1}C,\epsilon) \;=\; \varinjlim_{(B,\beta)} Q^n(B,\epsilon)\ .$$

It follows that the morphisms of ϵ-symmetric Γ- and L-groups

$$\sigma^{-1}\Gamma^* \;:\; \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon)\ ;\ (C,\phi) \mapsto \sigma^{-1}(C,\phi)$$

are also isomorphisms, and the localization exact sequence is given by Proposition 2.3. □

Hypothesis 2.6. *For the remainder of this section, we assume Hypothesis 2.1 and also that $A \longrightarrow \sigma^{-1}A$ is an injection.* □

As in Proposition 2.2 of [2] it follows that all the morphisms in σ are injections.

We shall now generalize the results of Ranicki [5] and Vogel [8] to prove that under Hypotheses 2.1,2.6 the relative L-groups $L^*(A,\sigma,\epsilon)$, $L_*(A,\sigma,\epsilon)$ in the L-theory localization exact sequences are the L-groups of $H(A,\sigma)$ with respect to the following duality involution.

Define the *torsion dual* of an (A,σ)-module M to be the (A,σ)-module

$$M\widehat{\ } \;=\; \mathrm{Ext}^1_A(M,A)\ ,$$

using the involution on A to define the left A-module structure. If M has f.g. projective A-module resolution

$$0 \longrightarrow P_1 \xrightarrow{s} P_0 \longrightarrow M \longrightarrow 0$$

with $s \in \sigma$ the torsion dual $M\widehat{\ }$ has the dual f.g. projective A-module resolution

$$0 \longrightarrow P_0^* \xrightarrow{s^*} P_1^* \longrightarrow M\widehat{\ } \longrightarrow 0$$

with $s^* \in \sigma$.

Proposition 2.7. *Let $M = \mathrm{coker}(s : P_1 \longrightarrow P_0)$, $N = \mathrm{coker}(t : Q_1 \longrightarrow Q_0)$ be (A,σ)-modules.*

(i) *The adjoint of the pairing*

$$M \times M\widehat{\ } \longrightarrow \sigma^{-1}A/A\ ;\ (g \in P_0, f \in P_1^*) \mapsto fs^{-1}g$$

defines a natural A-module isomorphism

$$M\widehat{\ } \longrightarrow \mathrm{Hom}_A(M,\sigma^{-1}A/A)\ ;\ f \mapsto (g \mapsto fs^{-1}g)\ .$$

(ii) *The natural A-module morphism*

$$M \longrightarrow M\widehat{\ }\widehat{\ }\ ;\ x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism.
(iii) *There are natural identifications*

$$\begin{aligned} M\otimes_A N &= \mathrm{Tor}_0^A(M,N) = \mathrm{Ext}^1_A(M\widehat{\ },N) = H_0(P\otimes_A Q)\ , \\ \mathrm{Hom}_A(M\widehat{\ },N) &= \mathrm{Tor}_1^A(M,N) = \mathrm{Ext}^0_A(M\widehat{\ },N) = H_1(P\otimes_A Q)\ . \end{aligned}$$

The functions

$$\begin{aligned} &M\otimes_A N\longrightarrow N\otimes_A M\ ;\ x\otimes y\mapsto y\otimes x\ , \\ &\mathrm{Hom}_A(M\widehat{\ },N)\longrightarrow \mathrm{Hom}_A(N\widehat{\ },M)\ ;\ f\mapsto f\widehat{\ } \end{aligned}$$

determine transposition isomorphisms

$$T\ :\ \mathrm{Tor}_i^A(M,N)\longrightarrow \mathrm{Tor}_i^A(N,M)\ (i=0,1)\ .$$

(iv) *For any finite subset $V=\{v_1,v_2,\dots,v_k\}\subset M\otimes_A N$ there exists an exact sequence of (A,σ)-modules*

$$0\longrightarrow N\longrightarrow L\longrightarrow \oplus_k M\widehat{\ }\longrightarrow 0$$

such that $V\subset \ker(M\otimes_A N\longrightarrow M\otimes_A L)$.

Proof. (i) Apply the snake lemma to the morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(P_0,A) \arrow[r] \arrow[d, "s^*"] & \mathrm{Hom}_A(P_0,\sigma^{-1}A) \arrow[r] \arrow[d, "s_1^*"] & \mathrm{Hom}_A(P_0,\sigma^{-1}A/A) \arrow[r] \arrow[d, "s_2^*"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_A(P_1,A) \arrow[r] & \mathrm{Hom}_A(P_1,\sigma^{-1}A) \arrow[r] & \mathrm{Hom}_A(P_1,\sigma^{-1}A/A) \arrow[r] & 0 \end{tikzcd}$$

with s^* injective, s_1^* an isomorphism and s_2^* surjective, to verify that the A-module morphism

$$M\widehat{\ } = \mathrm{coker}(s^*)\longrightarrow \mathrm{Hom}_A(M,\sigma^{-1}A/A) = \ker(s_2^*)$$

is an isomorphism.
(ii) Immediate from the identification

$$s^{**} = s\ :\ (P_0)^{**} = P_0\longrightarrow (P_1)^{**} = P_1\ .$$

(iii) Exercise for the reader.
(iv) Lift each $v_i\in M\otimes_A N$ to an element

$$v_i\in P_0\otimes_A Q_0 = \mathrm{Hom}_A(P_0^*,Q_0)\ (1\le i\le k)\ .$$

The A-module morphism defined by

$$u = \begin{pmatrix} s^* & 0 & 0 & \dots & 0 \\ 0 & s^* & 0 & \dots & 0 \\ 0 & 0 & s^* & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & t \end{pmatrix}\ :\ U_1 = (\oplus_k P_0^*)\oplus Q_1\longrightarrow U_0 = (\oplus_k P_1^*)\oplus Q_0$$

is in σ, so that $L = \mathrm{coker}(u)$ is an (A,σ)-module with a f.g. projective A-module resolution

$$0 \longrightarrow U_1 \xrightarrow{u} U_0 \longrightarrow L \longrightarrow 0 \ .$$

The short exact sequence of 1-dimensional f.g. projective A-module chain complexes

$$0 \longrightarrow Q \longrightarrow U \longrightarrow \oplus_k P^{1-*} \longrightarrow 0$$

is a resolution of a short exact sequence of (A,σ)-modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \oplus_k M\widehat{\ } \longrightarrow 0 \ .$$

The first morphism in the exact sequence

$$\mathrm{Tor}_1^A(M,\oplus_k M\widehat{\ }) \longrightarrow M\otimes_A N \longrightarrow M\otimes_A L \longrightarrow M\otimes_A(\oplus_k M\widehat{\ }) \longrightarrow 0$$

sends $1_i \in \mathrm{Tor}_1^A(M,\oplus_k M\widehat{\ }) = \oplus_k \mathrm{Hom}_A(M\widehat{\ },M\widehat{\ })$ to $v_i \in \ker(M\otimes_A N \longrightarrow M\otimes_A L)$. $\square$

Given an (A,σ)-module chain complex C define the *ϵ-symmetric* (resp. *ϵ-quadratic*) *torsion Q-groups* of C to be the $\mathbb{Z}_2$-hypercohomology (resp. $\mathbb{Z}_2$-hyperhomology) groups of the ϵ-transposition involution $T_\epsilon = \epsilon T$ on the $\mathbb{Z}$-module chain complex $\mathrm{Tor}_1^A(C,C) = \mathrm{Hom}_A(C\widehat{\ },C)$

$$\begin{aligned} Q^n_{\mathrm{tor}}(C,\epsilon) &= H^n(\mathbb{Z}_2;\mathrm{Tor}_1^A(C,C)) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Tor}_1^A(C,C))) \ , \\ Q_n^{\mathrm{tor}}(C,\epsilon) &= H_n(\mathbb{Z}_2;\mathrm{Tor}_1^A(C,C)) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\mathrm{Tor}_1^A(C,C))) \ . \end{aligned}$$

There are defined forgetful maps

$$\begin{aligned} &1+T_\epsilon \ : \ Q_n^{\mathrm{tor}}(C,\epsilon) \longrightarrow Q^n_{\mathrm{tor}}(C,\epsilon) \ ; \ \psi \mapsto (1+T_\epsilon)\psi \ , \\ &Q^n_{\mathrm{tor}}(C,\epsilon) \longrightarrow H_n(\mathrm{Tor}_1^A(C,C)) \ ; \ \phi \mapsto \phi_0 \ . \end{aligned}$$

The element $\phi_0 \in H_n(\mathrm{Tor}_1^A(C,C))$ is a chain homotopy class of A-module chain maps $\phi_0 : C^{n-\widehat{\ }} \longrightarrow C$.

An *n-dimensional ϵ-symmetric complex over* (A,σ) (C,ϕ) is a bounded (A,σ)-module chain complex C together with an element $\phi \in Q^n_{\mathrm{tor}}(C,\epsilon)$. The complex (C,ϕ) is *Poincaré* if the A-module chain maps $\phi_0 : C^{n-\widehat{\ }} \longrightarrow C$ are chain equivalences.

Example 2.8. A 0-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over (A,σ) is essentially the same as a nonsingular ϵ-symmetric linking form (M,λ) over (A,σ), with $M = (C_0)\widehat{\ }$ an (A,σ)-module and

$$\lambda = \phi_0 \ : \ M\times M \longrightarrow \sigma^{-1}A/A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M\widehat{\ } \ ; \ x \mapsto (y \mapsto \lambda(x,y))$$

is an A-module isomorphism.

$\square$

The *n-dimensional torsion ϵ-symmetric L-group* $L^n_{\mathrm{tor}}(A,\sigma,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes (C,ϕ) over (A,σ), with C n-dimensional. In particular, $L^0_{\mathrm{tor}}(A,\sigma,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric linking forms over (A,σ).

Similarly in the ϵ-quadratic case, with torsion L-groups $L_n^{\mathrm{tor}}(A,\sigma,\epsilon)$. The ϵ-quadratic torsion L-groups are 4-periodic

$$L_n^{\mathrm{tor}}(A,\sigma,\epsilon) \;=\; L_{n+2}^{\mathrm{tor}}(A,\sigma,-\epsilon) \;=\; L_{n+4}^{\mathrm{tor}}(A,\sigma,\epsilon)\ .$$

Theorem 2.9. *If $A \longrightarrow \sigma^{-1}A$ is injective the relative L-groups in the localization exact sequences of Proposition 2.3*

$$\cdots \longrightarrow L^n(A,\epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \cdots$$
$$\cdots \longrightarrow L_n(A,\epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots$$

are the torsion L-groups

$$\begin{aligned} L^*(A,\sigma,\epsilon) \;&=\; L^*_{\mathrm{tor}}(A,\sigma,\epsilon)\ ,\\ L_*(A,\sigma,\epsilon) \;&=\; L_*^{\mathrm{tor}}(A,\sigma,\epsilon)\ . \end{aligned}$$

Proof. For any bounded (A,σ)-module chain complex T there exists a bounded f.g. projective A-module chain complex C with a homology equivalence $C \longrightarrow T$. Working as in [8] there is defined a distinguished triangle of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\Sigma\mathrm{Tor}_1^A(T,T) \longrightarrow C\otimes_A C \longrightarrow T\otimes_A T \longrightarrow \Sigma^2\mathrm{Tor}_1^A(T,T)$$

with $\mathbb{Z}_2$ acting by the ϵ-transposition T_ϵ on the $\mathbb{Z}$-module chain complex $\mathrm{Tor}_1^A(T,T)$ and by the $(-\epsilon)$-transpositions $T_{-\epsilon}$ on $C\otimes_A C$ and $T\otimes_A T$, inducing long exact sequences

$$\cdots \longrightarrow Q^n_{\mathrm{tor}}(T,\epsilon) \longrightarrow Q^{n+1}(C,-\epsilon) \longrightarrow Q^{n+1}(T,-\epsilon) \longrightarrow Q^{n-1}_{\mathrm{tor}}(T,\epsilon) \longrightarrow \cdots$$
$$\cdots \longrightarrow Q_n^{\mathrm{tor}}(T,\epsilon) \longrightarrow Q_{n+1}(C,-\epsilon) \longrightarrow Q_{n+1}(T,-\epsilon) \longrightarrow Q_{n-1}^{\mathrm{tor}}(T,\epsilon) \longrightarrow \cdots\ .$$

Passing to the direct limits over all the bounded (A,σ)-module chain complexes U with a homology equivalence $\beta : T \longrightarrow U$ use Proposition 2.7 (iv) to obtain

$$\begin{aligned} \varinjlim_{(U,\beta)} Q^{n+1}(U,-\epsilon) \;&=\; 0\ ,\\ \varinjlim_{(U,\beta)} Q_{n+1}(U,-\epsilon) \;&=\; 0 \end{aligned}$$

and hence

$$\begin{aligned} \varinjlim_{(U,\beta)} Q^n_{\mathrm{tor}}(U,\epsilon) \;&=\; Q^{n+1}(C,-\epsilon)\ ,\\ \varinjlim_{(U,\beta)} Q_n^{\mathrm{tor}}(U,\epsilon) \;&=\; Q_{n+1}(C,-\epsilon)\ . \end{aligned}$$

$\square$

Remark 2.10. The identification $L_*(A,\sigma,\epsilon) = L^{\mathrm{tor}}_*(A,\sigma,\epsilon)$ for noncommutative $\sigma^{-1}A$ was first obtained by Vogel [8].

□

References

[1] Amnon Neeman, *Triangulated categories*, Annals of Math. Studies, vol. **148**, Princeton University Press (2001)

[2] ——, *Noncommutative localisation in algebraic K-theory II.*, Advances in Mathematics 213, 785–819 (2007)

[3] —— and Andrew Ranicki, *Noncommutative localisation in algebraic K-theory I.* Geometry and Topology 8, 1385–1425 (2004)

[4] ——, —— and Aidan Schofield, *Representations of alegbras as universal localizations*, Math. Proc. Camb. Phil. Soc. 136, 105–117 (2004)

[5] Andrew Ranicki, *Exact sequences in the algebraic theory of surgery*, Mathematical Notes **26**, Princeton (1981)

[6] ——, *High dimensional knot theory*, Springer Monograph, Springer (1998)

[7] ——, *Noncommutative localization in topology*, Proc. 2002 ICMS conference on Noncommutative Localization in Algebra and Topology, LMS Lecture Notes 330, CUP, 81–102 (2006)

[8] Pierre Vogel, *Localization in algebraic L-theory*, Proc. 1979 Siegen Topology Symposium, Lecture Notes in Mathematics **788**, Springer Verlag, 1980, pp. 482–495.

[9] ——, *On the obstruction group in homology surgery*, Publ. Math. I.H.E.S. **55** (1982), 165–206.

A.R. : School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
SCOTLAND, UK
E-mail address: a.ranicki@ed.ac.uk